

ASYMMETRY IN HILBERT'S FOURTH PROBLEM

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To Fred Almgren, in memoriam

ABSTRACT. In the asymmetric setting, Hilbert's fourth problem asks to construct and study all (non-reversible) projective Finsler metrics: Finsler metrics defined on open, convex subsets of real projective n -space for which geodesics lie on projective lines. While asymmetric norms and Funk metrics provide many examples of essentially non-reversible projective metrics defined on *proper* convex subsets of projective n -space, it is shown that *any projective Finsler metric defined on the whole projective n -space is the sum of a reversible projective metric and an exact 1-form*. The result admits a wide generalization to path geometries other than projective lines in real projective spaces.

1. INTRODUCTION

In the Summer of 1996 Eric Grinberg and Fred Almgren organized a very small and very informal meeting on Hilbert's fourth problem at the Institute of Advanced Studies. In the discussion after the talks Almgren asked whether I had considered asymmetric variational problems where the Lagrangian function does not satisfy $L(x, v) = L(x, -v)$ and, more precisely, whether every non-reversible Lagrangian on the plane whose extremals are straight lines is the sum of a reversible Lagrangian and the differential of a smooth function on the plane. I answered that any asymmetric norm in the plane provides an example that is not of this form and forgot about Almgren's question for the next fourteen years. In 2010 Gautier Berck surprised me with the following result:

Theorem (Berck). *A projective Finsler metric defined on the whole real projective plane is the sum of a reversible projective Finsler metric and an exact 1-form.*

Berck's one-page proof is simple and elegant. However, it is also purely two-dimensional and applies only to projective Finsler metrics. The aim of this short note is to show through a different circle of ideas that the result

1991 *Mathematics Subject Classification*. 53C65, 53C60, 53C22.

Key words and phrases. Finsler manifold, geodesics, path geometry, Brunn-Minkowski inequality, weak Blaschke conjecture.

This work was partially supported by Programme ANR Blanc - SIMI 1 - ANR 2012.

also holds in higher dimensions and for many systems of paths other than projective lines.

In order to state the main result of this note in a clear and succinct form, let us recall that a Finsler metric is *geodesically reversible* if every oriented geodesic can be reparametrized as a geodesic with the reverse orientation, and let us say that a Finsler metric is a *Zoll metric* if all of its geodesics are closed and of the same length.

Main Result. *Let M be a manifold diffeomorphic to a compact rank-one symmetric space. If L is a geodesically reversible Zoll metric on M , then L is the sum of a reversible Zoll metric and an exact 1-form.*

In the particular case of projective Finsler metrics on $\mathbb{R}P^n$ or S^n , this theorem completely solves the asymmetric version of Hilbert's fourth problem by stating that, in fact, *there are no interesting non-reversible projective metrics on these spaces*. Notice however, that there are many interesting non-reversible projective metrics such as asymmetric norms or Funk metrics if we look at proper convex subsets of projective space.

2. PRELIMINARIES

In this note a function $F : TM \rightarrow [0.. \infty)$ will be called a *Finsler metric* on the manifold M if it is smooth outside the zero section and its restriction to each tangent space $T_x M$ is a (possibly) asymmetric, quadratically convex norm F_x . Equivalently, the restriction of F to each tangent space $T_x M$ is the support function of a quadratically convex body $D_x^* M \subset T_x^* M$ that contains the origin in its interior. Finsler metrics for which all norms F_x , $x \in M$, are symmetric will be called *reversible Finsler metrics*. The reader may wonder why it is necessary to use two different terms—symmetry and reversibility—for the same concept, but this usage solves the problem of how to call the Finsler analogues of Riemannian symmetric spaces. Moreover, Finsler geometry is part of the geometry of Hamiltonian systems and the term reversible or *time-reversible* Hamiltonian is well-established.

The proof of our main result turns out to be surprisingly simple. Its key technical ingredients are the Brunn-Minkowski inequality and the Weinstein-Yang-Reznikov solution of what was then called the *weak Blaschke conjecture*. The key conceptual ingredient is the *Holmes-Thompson volume*.

Definition. Let (M, F) be an n -dimensional Finsler manifold and let $H : T^*M \rightarrow [0.. \infty)$ be the Hamiltonian function obtained from F by the Legendre transform. The Holmes-Thompson volume of (M, F) is the symplectic volume of the unit co-disc bundle $H \leq 1$ divided by the volume of the Euclidean unit ball of dimension n .

We may also define the Holmes-Thompson volume of (M, F) as the contact volume of the unit co-sphere bundle $H = 1$ divided by the volume of the Euclidean unit ball of dimension n . Note that the Holmes-Thompson

volume of a Riemannian manifold is its usual volume, which can also be defined as the Hausdorff measure of its associated metric. However, for more general normed spaces and reversible Finsler manifolds the two notions of volume do not coincide. In (non-reversible) Finsler manifolds there does not seem to be a reasonable generalization of the Hausdorff measure. The reader is referred to [6] and [1] for more information about the Holmes-Thompson volume and other notions of volume on normed and Finsler spaces. For the purposes of this paper, we need only remark that the symplectic character of the Holmes-Thompson volume makes it the volume of choice to trivially extend Riemannian results that only depend on the symplectic properties of the geodesic flow. One such result is the proof of the weak Blaschke conjecture by A. Weinstein, C.T. Yang, and A. Reznikov (see [7], [8, 9], and [4, 5]). The following statement summarizes their results and puts them in a Finsler-geometric setting.

Theorem (Weak Blaschke Conjecture). *Let F and G be two Zoll metrics on a manifold M diffeomorphic to a compact rank-one symmetric space. If the lengths of the (prime) closed geodesics of (M, F) and (M, G) coincide, then their Holmes-Thompson volumes coincide as well.*

3. PROOF OF THE MAIN THEOREM

Let $a : TM \rightarrow TM$ be the map that sends each tangent vector v to its opposite $-v$. The main result follows from a comparison between the properties of the Finsler metric L and those of its symmetrization $\bar{L} := (L + L \circ a)/2$. First we show that symmetrization increases volume:

Lemma. *The Holmes-Thompson volume of a compact Finsler manifold (M, L) is less than or equal to that of its symmetrization (M, \bar{L}) . Moreover, the volumes are equal if and only if L equals its symmetrization modulo the addition of a 1-form. In other words, $L = \bar{L} + \beta$, where β is a 1-form on M .*

Proof. For each $x \in M$ denote by $D_x^*M \subset T^*M$ the convex body supported by the restriction of L to T_xM . Note that the restriction of \bar{L} to T_xM is the support function of the symmetrized body $(D_x^*M + -D_x^*M)/2$. A well-known consequence of the Brunn-Minkowski inequality (see, for example, [2]) is that the volume of a convex body K is less than or equal to that of its symmetrization $(K + -K)/2$ with equality if and only if K is symmetric with respect to a (unique) point p in its interior. Applying the theorem at every cotangent space T_xM , $x \in M$, proves the lemma. Note that the 1-form will be smooth if the bodies D_x^*M depend smoothly on the base point, which is the case. \square

Definition. Two Finsler metrics are *projectively equivalent* if they have the same geodesics up to orientation-preserving reparametrization.

Lemma. *A geodesically reversible Finsler metric L is projectively equivalent to its symmetrization $\bar{L} := (L + L \circ a)/2$.*

Proof. Remark only that projectively equivalent Finsler metrics form a convex cone and that L is geodesically reversible if and only if L and $L \circ a$ are projectively equivalent. \square

Lemma (Crampin [3]). *If (M, F) is a reversible Finsler manifold and β is a 1-form on M , the Lagrangian $F + \beta$ is not geodesically reversible unless β is closed.*

Proof of the Main Result. Since L is geodesically reversible, it is projectively equivalent to its symmetrization \bar{L} and, therefore, L and \bar{L} are two Zoll metrics such that the lengths of their (prime) closed geodesics coincide. By the solution of the Weak Blaschke Conjecture, the Holmes-Thompson volume of (M, L) equals that of (M, \bar{L}) . Applying the equality case of the Brunn-Minkowski inequality, we conclude that L equals its symmetrization modulo the addition of a 1-form β . Crampin's Lemma now implies that β is closed and—given the topology of M —exact. \square

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